Role of Information in Pricing Default-Sensitive Contingent Claims

Marta Leniec

Joint work with Monique Jeanblanc

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Set-up

- Let $(\Omega, \mathcal{A}, \mathbb{A} = (\mathcal{A}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where:
  - $\mathbb{A}$ is the complete, right-continuous **filtration** generated by a 2-dimensional Brownian motion $(B, B^{\perp})$,
  - $\mathbb{P}$ is the **historical** probability measure.
- Define $W = (W_t)_{t\geq 0}$ by

$$W_t = \rho B_t + \sqrt{1-\rho^2} B_t^{\perp}.$$

  Then W is a an $(\mathbb{A}, \mathbb{P})$-Brownian motion correlated with B with the correlation coefficient $\rho \in (-1, 1)$.

- Let $\mathbb{F}^X = (\mathcal{F}_t^X)_{t\geq 0}$ denote the complete, right-continuous **filtration** generated by a process X.
- The filtration $\mathbb{A}$ coincides with the filtration $\mathbb{F}^W \vee \mathbb{F}^B$, generated by the 2-dimensional correlated Brownian motion (W, B).

Model

Let:

- S, the **stock price**, be a strictly positive $\mathbb{F}^W$-adapted diffusion satisfying

$$dS_t = S_t(\mu(S_t)dt + \sigma(S_t)dW_t), \quad S_0 = s_0,$$

- V, the **value of the firm**, be an $\mathbb{F}^B$-adapted diffusion starting at $V_0 = v_0$ which satisfies $\mathcal{F}_t^V = \mathcal{F}_t^B$,

- τ, the **default of the firm**, be a strictly positive, finite $\mathbb{F}^B$-stopping time.

We assume that the **law** of τ is equivalent to the Lebesgue measure, i.e., $\mathbb{P}(\tau \in du) \sim du$. Consequently, there exists a strictly positive **density** function g such that

$$\mathbb{P}(\tau > u) = \int_u^\infty g(s)ds \quad \forall u \geq 0.$$

Moreover, we suppose that

$$G_t = \mathbb{P}(\tau > t | \mathcal{F}_t^W) > 0.$$

Default-free Market

We consider a **financial market** with:

- the **stock** $S = (S_t)_{t \geq 0}$, where

$$dS_t = S_t(\mu(S_t)dt + \sigma(S_t)dW_t), \quad S_0 = s_0,$$

- the **bank account** $B = (B_t)_{t \geq 0}$ satisfying

$$dB_t = r_t B_t dt, \quad B_0 = 1.$$

We assume that:

- the **risk-free** $r_t \equiv 0$,

- $\theta = (\theta_t)_{t \geq 0}$**, where** $\theta_t = \frac{\mu(S_t)}{\sigma(S_t)}$, **satisfies the Novikov's condition.**

Default-Sensitive Contingent Claim

Definition

A **default-sensitive contingent claim** is a random variable of the form

$$\Psi^\tau = Y_T^1 \mathbb{I}_{\tau > T} + Y_T^2(\tau)\mathbb{I}_{\tau \le T},$$

where Y_T^1 and $Y_T^2(u)$ are $\mathcal{F}_T^W$-measurable random variables.

Example

Default-sensitive European call option:

$$\Psi^\tau = \mathbb{I}_{\tau > T}(S_T - K)^+ + \mathbb{I}_{\tau \le T} h(\tau)(S_T - K)^+$$

Definition

A **fair price** at time $t \in [0, T]$ of the **contingent claim** Ψ^τ is the conditional expectation of Ψ^τ with respect to $\mathcal{K}_t$ under one of the pricing measures $\mathbb{Q}$, i.e.,

$$\mathbb{C}_t = \mathbb{E}_{\mathbb{Q}}(\Psi^\tau | \mathcal{K}_t).$$

Additional Information

The filtration $\mathbb{K}$ represents:

- either the **regular investor's information**

$$\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau \wedge t) \quad \forall t \geq 0,$$

- or the **strong information**

$$\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau) \quad \forall t \geq 0,$$

- or the **full information**

$$\mathcal{K}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^B \vee \sigma(\tau) \quad \forall t \geq 0.$$

Hypothesis (Jacod's Hypothesis)

The $\mathbb{F}^W$-**(regular) conditional law** *of* τ *is equivalent to the law of* τ, *i.e.,*

$$\mathbb{P}(\tau \in du|\mathcal{F}_t^W) \sim \mathbb{P}(\tau \in du) \quad \mathbb{P} - \textit{a.s.} \quad \forall t \geq 0.$$

Lemma

There exists a family of strictly positive $(\mathbb{F}^W, \mathbb{P})$*- martingales* $(p(u))_{u \geq 0}$, *called the* $\mathbb{F}^W$-**conditional density** *of* τ, *such that*

$$\mathbb{P}(\tau > u|\mathcal{F}_t^W) = \int_u^\infty p_t(s)g(s)ds, \quad \mathbb{P} - a.s. \quad \forall t \geq 0.$$

Fact

For every $u \geq 0$, *there exists a process* $\beta(u)$ *such that*

$$dp_t(u) = p_t(u)\beta_t(u)dW_t, \quad p_0(u) = 1.$$

$\mathbb{F}^W \vee \mathbb{F}^B$-(regular) conditional law of τ

Theorem

The $\mathbb{F}^W \vee \mathbb{F}^B$-**(regular) conditional law** *of* τ *is given by*

$$\mathbb{P}(\tau > u | \mathcal{F}_t^W \vee \mathcal{F}_t^B) = \begin{cases} \mathbb{I}_{\tau > t} \mathbb{P}(\tau > u | \mathcal{F}_t^B), & u > t, \\ \mathbb{I}_{\tau > u}, & u \leq t. \end{cases}$$

Yor's Method

- Let us denote

$$P_t(u) = \mathbb{P}(\tau > u | \mathcal{F}_t^W \vee \mathcal{F}_t^B) = \mathbb{P}(\tau > u | \mathcal{F}_t^B).$$

- The process $P(u) = (P_t(u))$ is an $(\mathbb{F}^B, \mathbb{P})$-martingale. Hence,

$$P_t(u) = P_0(u) + \int_0^t Q_s(u) dB_s.$$

- There exists a family of measures $P(dx) = (P_t(dx))_{t \geq 0}$ such that

$$P_t(u) = \int_u^\infty P_t(dx).$$

Example

Take S and V which satisfy the following:

$$dS_t = S_t \sigma \rho dW_t, \quad S_0 = s_0,$$
$$dV_t = V_t \sigma dB_t, \quad V_0 = v_0,$$

where

$$d\langle W, B \rangle_t = \rho dt$$

and

$$\tau = \inf\{t \geq 0 : V_t \leq a\}.$$

Define $P(u)$ by

$$P_t(u) = \mathbb{P}(\tau > u | \mathcal{F}_t^W \vee \mathcal{F}_t^B) = \mathbb{P}(\tau > u | \mathcal{F}_t^B).$$

Then,

$$P_t(u) = \begin{cases} \mathbb{I}_{\tau > t} \int_u^\infty \frac{\ln(\frac{V_t}{a})}{\sigma(s-t)^{\frac{3}{2}} \sqrt{2\pi}} \exp\{-\frac{(X_t^2(s)}{2}\} ds, & u > t, \\ \mathbb{I}_{\tau > u}, & u \leq t, \end{cases}$$

and for $t < \tau$

$$Q_t(u) = \mathbb{I}_{t<u} \Big(\frac{2}{\sqrt{2\pi(u-t)}} \exp\{-\frac{X_t^2(u)}{2}\} - \frac{V_t}{a} \sigma \Phi(-X_t(u) - \sigma\sqrt{u-t}) \Big),$$

where

$$X_t(u) = \frac{\ln(\frac{V_t}{a}) - \frac{1}{2}\sigma^2(u-t)}{\sigma\sqrt{u-t}}.$$

Yor's Method

Hypothesis (Yor's Hypothesis)

There exists an $\mathbb{F}^W \vee \mathbb{F}^B$*-predictable family of measures* $Q(dx) = (Q_t(dx))_{t\geq 0}$ *such that*

$$Q_t(u) = \int_u^\infty Q_t(dx)$$

and the measure $Q_t(dx)$ *is* **absolutely continuous** *with respect to* $P_t(dx)$.
Moreover, the process $\zeta(x)$ *defined by*

$$Q_t(dx) = \zeta_t(x) P_t(dx)$$

satisfies

$$\int_0^t |\zeta_s(\tau)| ds < \infty \quad \mathbb{P} - \textit{a.s.} \quad \forall t \geq 0.$$

Example

The processes $P(u)$, $Q(u)$ and $\zeta(\tau) = (\zeta_t(\tau))_{t\geq 0}$, where

$$\zeta_t(\tau) = \mathbb{I}_{\tau > t} \frac{Q_t(du)}{P_t(du)}|_{u=\tau} = \mathbb{I}_{\tau > t}\Big(\frac{1}{B_\tau - B_t - \frac{1}{2}\sigma(\tau - t)} - \frac{B_\tau - B_t}{\tau - t}\Big),$$

satisfy the **Yor's Hypothesis.**

- Decomposition of W in $\mathbb{K}$:

Filtration $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$	Decomposition of W
$\mathcal{F}_t^W \vee \sigma(\tau \wedge t)$	$W_t = W_t^{\mathbb{K}} + \int_0^{t\wedge\tau} \frac{\Xi_s}{G_s} ds + \int_{t\wedge\tau}^t \beta_s(\tau) ds$
$\mathcal{F}_t^W \vee \sigma(\tau)$	$W_t = W_t^{\mathbb{K}} + \int_0^t \beta_s(\tau) ds$
$\mathcal{F}_t^W \vee \mathcal{F}_t^B \vee \sigma(\tau)$	$W_t = W_t^{\mathbb{K}} + \rho \int_0^t \zeta_s(\tau) ds$

Semi-martingale Decomposition of S

- Semi-martingale decomposition of S in $\mathbb{K}$:

$$dS_t = S_t\Big(A_t dt + \sigma(S_t) dW_t^{\mathbb{K}}\Big)$$

Filtration $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$	Decomposition of $A = (A_t)_{t\geq 0}$
$\mathcal{F}_t^W \vee \sigma(\tau \wedge t)$	$A_t = \mu(S_t) + \sigma(S_t)\Big(\mathbb{I}_{t\leq\tau}\frac{\Xi_t}{G_t} + \mathbb{I}_{t>\tau}\beta_t(\tau)\Big)$
$\mathcal{F}_t^W \vee \sigma(\tau)$	$A_t = \mu(S_t) + \sigma(S_t)\beta_t(\tau)$
$\mathcal{F}_t^W \vee \mathcal{F}_t^B \vee \sigma(\tau)$	$A_t = \mu(S_t) + \rho\sigma(S_t)\zeta_t(\tau)$

Predictable Representation Theorem

- Any strictly positive $(\mathbb{K}, \mathbb{P})$-(local) martingale N has the following decomposition:

Filtration $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$	Decomposition of $N = (N_t)_{t\geq 0}$
$\mathcal{F}_t^W \vee \sigma(\tau \wedge t)$	$dN_t = N_{t-}\left(a_t dW_t^{\mathbb{K}} + b_t dM_t\right)$
$\mathcal{F}_t^W \vee \sigma(\tau)$	$dN_t(\tau) = N_t(\tau) a_t(\tau) dW_t^{\mathbb{K}}$
$\mathcal{F}_t^W \vee \mathcal{F}_t^B \vee \sigma(\tau)$	$dN_t(\tau) = N_t(\tau)\left(a_t(\tau) dW_t^{\mathbb{K}} + b_t(\tau) dW_t^{\mathbb{K},\perp}\right)$

Filtration $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$	Radon-Nikodym density $Z = (Z_t)_{t\geq 0}$
$\mathcal{F}^W_t \vee \sigma(\tau \wedge t)$	$Z_t = \mathcal{E}\left(-\int_0^{\cdot}(\theta_s + \gamma_s)dW^{\mathbb{K}}_s\right)_t \exp\{-\int_0^{t\wedge\tau} b_s\lambda_s ds + \ln(1+b_\tau)\mathbb{I}_{t>\tau}\}$
$\mathcal{F}^W_t \vee \sigma(\tau)$	$Z_t(\tau) = Z_0(\tau)\mathcal{E}\left(-\int_0^{\cdot}(\theta_s + \beta_s(\tau))dW^{\mathbb{K}}_s\right)_t$
$\mathcal{F}^W_t \vee \mathcal{F}^B_t \vee \sigma(\tau)$	$Z_t(\tau) = Z_0(\tau)\mathcal{E}\left(-\int_0^{\cdot}\left(\theta_s + \rho\zeta_s(\tau)\right)dW^{\mathbb{K}}_s\right)_t \mathcal{E}\left(\int_0^{\cdot} b_s(\tau)dW^{\mathbb{K},\perp}_s\right)_t$

Proposition

The process Z is a $(\mathbb{K},\mathbb{P})$-martingale and the probability measure $\mathbb{Q}$ defined by

$$d\mathbb{Q}|_{\mathcal{K}_t} = Z_t d\mathbb{P}|_{\mathcal{K}_t} \quad \forall t \geq 0$$

belongs to the set

$$\{\mathbb{Q} : \mathbb{Q} \overset{loc}{\sim} \mathbb{P}, \quad S \text{ is a } (\mathbb{K},\mathbb{Q}) - \text{(local) martingale}\}.$$

Price in $\mathbb{F}^W \vee \sigma(\tau)$

Let $\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau)$. The **Radon-Nikodym** density $Z = (Z_t)_{t \geq 0}$ is given by

$$Z_t(\tau) = Z_0(\tau)\mathcal{E}\left(-\int_0^\cdot (\theta_s + \beta_s(\tau))dW_s^{\mathbb{K}}\right)_t.$$

Proposition

The **price** *of the* **default-sensitive contingent claim**

$$\Psi^\tau = Y_T^1 \mathbb{I}_{\tau > T} + Y_T^2(\tau)\mathbb{I}_{\tau \leq T},$$

where Y_T^1 and $Y_T^2(u)$ are $\mathcal{F}_T^W$-measurable random variables, is **uniquely** *given by*

$$\boxed{\mathbb{C}_t(\tau) = \mathbb{I}_{\tau > T}\frac{\mathbb{E}_{\mathbb{P}}(Y_T^1 m_T | \mathcal{F}_t^W)}{m_t} + \mathbb{I}_{\tau \leq T}\frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u) m_T | \mathcal{F}_t^W)_{|u=\tau}}{m_t}}$$

where $m = (m_t)_{t \geq 0}$ satisfies

$$m_t = \mathcal{E}\left(-\int_0^\cdot \theta_s dW_s\right)_t \quad \forall t \geq 0.$$

Price in $\mathbb{F}^W \vee \mathbb{F}^B \vee \sigma(\tau)$

Let $\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau)$. The **Radon-Nikodym** density $Z = (Z_t)_{t\geq 0}$ is given by

$$Z_t(\tau) = Z_0(\tau)\mathcal{E}\left(-\int_0^{\cdot}\Big(\theta_s + \rho\zeta_s(\tau)\Big)dW_s^{\mathbb{K}}\right)_t \mathcal{E}\left(\int_0^{\cdot} b_s(\tau)dW_s^{\mathbb{K},\perp}\right)_t .$$

Proposition

The **price** *of the* **default-sensitive contingent claim**

$$\Psi^\tau = Y_T^1 \mathbb{I}_{\tau > T} + Y_T^2(\tau)\mathbb{I}_{\tau \leq T},$$

where Y_T^1 and $Y_T^2(u)$ are $\mathcal{F}_T^W$-measurable random variables, is **uniquely** *given by*

$$\boxed{\mathbb{C}_t(\tau) = \mathbb{I}_{T<\tau}\frac{\mathbb{E}_{\mathbb{P}}(Y_T^1 L_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)} + \mathbb{I}_{T\geq\tau}\frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u) L_T(u)|\mathcal{F}_t^1 \vee \mathcal{F}_t^2)|_{u=\tau}}{L_t(\tau)}}$$

where $L(\tau) = (L_t(\tau))_{t\geq 0}$ satisfies $L_t(\tau) = \mathcal{E}\left(-\int_0^{\cdot}\Big(\theta_s + \rho\zeta_s(\tau)\Big)dW_s^{\mathbb{K}}\right)_t \Pi_t(u)|_{u=\tau}$ and $\Pi_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t^B)$.

We **choose** the process b in

$$Z_t = \mathcal{E}\left(-\int_0^{\cdot} (\theta_s + \gamma_s) dW_s^{\mathbb{K}}\right)_t \exp\{-\int_0^{t\wedge\tau} b_s \lambda_s ds + \ln(1 + b_\tau)\mathbb{I}_{t>\tau}\},$$

which satisfies the following conditions:

- b is $\mathbb{F}^W$-predictable,
- $\mathbb{E}_{\mathbb{P}}(b_\tau^2) < \infty$,
- $b_t > -1$ for all $t \geq 0$,
- $\mathbb{E}_{\mathbb{P}}(Z_t) = 1$.

f-divergence, where $f(x) = -\ln(x)$

We **choose**

$$\mathbb{Q}^* \in \{\mathbb{Q} : \mathbb{Q} \overset{loc}{\sim} \mathbb{P}, \quad S \text{ is a } (\mathbb{K}, \mathbb{Q}) - \text{(local) martingale}\},$$

such that

$$\mathbb{E}_{\mathbb{P}}\left(-\ln\left(\frac{d\mathbb{Q}^*}{d\mathbb{P}}|_{\mathcal{K}_T}\right)\right) = \inf_{\mathbb{Q}\in\mathcal{M}_{\mathbb{P}}(\mathbb{K})} \mathbb{E}_{\mathbb{P}}\left(-\ln\left(\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{K}_T}\right)\right),$$

where $\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau \wedge t)$.

We solve

$$\mathbb{E}_{\mathbb{P}}(-\ln(Z_T^*)) = \inf_{b\in\Delta} \mathbb{E}_{\mathbb{P}}(-\ln(Z_T)),$$

where

$$Z_T = \mathcal{E}\left(-\int_0^{\cdot} (\theta_s + \gamma_s) dW_s^{\mathbb{K}}\right)_T \exp\{-\int_0^{T\wedge\tau} b_s \lambda_s ds + \ln(1 + b_\tau)\mathbb{I}_{T>\tau}\}.$$

Minimal Martingale Measure in $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$, where $\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau \wedge t)$

Proposition

The **optimization problem** *is solved by $b_t^* \equiv 0$ for all $t \geq 0$, i.e.,*

$$Z_T^* = \mathcal{E}\left(-\int_0^\cdot \big(\theta_s + \frac{\Xi_s}{G_s}\mathbb{I}_{s\leq\tau} + \beta_s(\tau)\mathbb{I}_{s>\tau}\big)dW_s^{\mathbb{K}}\right)_T.$$

More precisely, we have

$$d\mathbb{Q}^*|_{\mathcal{K}_T} = \mathcal{E}\left(-\int_0^\cdot \big(\theta_s + \frac{\Xi_s}{G_s}\mathbb{I}_{s\leq\tau} + \beta_s(\tau)\mathbb{I}_{s>\tau}\big)dW_s^{\mathbb{K}}\right)_T d\mathbb{P}|_{\mathcal{K}_T}.$$

Price in $\mathbb{K} = (\mathcal{K}_t)_{t\geq 0}$, where $\mathcal{K}_t = \mathcal{F}_t^W \vee \sigma(\tau \wedge t)$

Proposition

The **price** *of the* **default-sensitive contingent claim**

$$\Psi^\tau = Y_T^1 \mathbb{I}_{\tau > T} + Y_T^2(\tau)\mathbb{I}_{\tau \leq T},$$

where Y_T^1 and $Y_T^2(u)$ are $\mathcal{F}_T^W$-measurable random variables, is given by

$$\mathbb{C}_t = \mathbb{I}_{t<\tau} \frac{\mathbb{E}_{\mathbb{P}}\left(\int_t^T Y_T^2(u)\hat{Z}_T^*(u)p_T(u)g(u)du + Y_T^1\tilde{Z}_T^* G_T | \mathcal{F}_t^1\right)}{G_t\tilde{Z}_t^*}$$

$$+\mathbb{I}_{t\geq\tau}\frac{\mathbb{E}_{\mathbb{P}}(Y_T^2(u)\hat{Z}_T^*(u)p_T(u)|\mathcal{F}_t^1)|_{u=\tau}}{p_t(\tau)\hat{Z}_t^*(\tau)}$$

where

$$Z_t^* = \tilde{Z}_t^*\mathbb{I}_{t\leq\tau} + \hat{Z}_t^*\mathbb{I}_{t>\tau}.$$

Conclusions

- In the presented method, one of the two **hypotheses (Jacod's or Yor's)** has to be satisfied.

- If one of the **hypotheses** is satisfied, then the **stock price** remains a **semi-martingale** in the **enlarged** filtration and the well-known results for the decomposition of a Brownian motion in the enlarged filtration may be applied.

- In the case of the **strong information** (initial enlargement of $\mathbb{F}^W$), the set of equivalent martingale measures is **infinite** but it does not imply incompleteness of the market because the σ-algebra $\mathcal{K}_0$ is **not trivial**. Moreover, the prices are **unique.**

- In the case of the **full information** (initial enlargement of $\mathbb{F}^W \vee \mathbb{F}^B$), the set is again infinite and the prices are **unique.**

- In the case of the **regular investor** (progressive enlargement of $\mathbb{F}^W$), the set of equivalent martingale measures is also **infinite** and the prices depend on the choice of a measure. One may use f-**divergence** approach to choose one of them.

Thank you for your attention.

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